§1.3 Lie Superalgebras

Generators of the superconformal group belong to Lie Super Algebra (LSA)

This is an algebra A with a $\mathbb{Z}_2 = \mathbb{Z}/2$ grading

$$A = \bigoplus_{\alpha \in \mathbb{Z}_2} A_\alpha = A_{\bar{0}} \oplus A_{\bar{1}}$$

such that $A_\alpha A_\beta \subseteq A_{\alpha+\beta}$

Elements of $A_{\bar{0}}$ are called "even", those of $A_{\bar{1}}$ "odd"

Define commutator:

$$[a, b] = ab - (-1)^{\deg(a)\deg(b)}\, ba$$

where $a, b \in A$ and $\deg \in \mathbb{Z}_2$

A "Lie superalgebra" is a superalgebra $G = G_{\bar{0}} \oplus G_{\bar{1}}$ with operation $[\ ,\ ]$ satisfying

$$[a, b] = -(-1)^{(\deg a)(\deg b)}[b, a]$$

$$[a, [b, c]] = [[a, b], c] + (-1)^{(\deg a)(\deg b)}[b, [a, c]]$$

(Jacobi identity)

We have: even $\underset{[\;,\;]}{\underset{\uparrow}{\times}}$ even = even, odd × odd = even

odd × even = odd, even × odd = odd

Construction:

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a $\mathbb{Z}_2$-graded vector space $\longrightarrow$ End V is endowed with $\mathbb{Z}_2$ grading

$\rightarrow$ associative superalgebra

notation: $\mathrm{End}(V) = \ell(V) = \ell(m,n)$,

where $m = \dim V_{\bar{0}}$, $n = \dim V_{\bar{1}}$

$$\ell(V) = \underset{\substack{\| \\ G_0}}{\ell(V)_0} \oplus \underset{\substack{\| \\ G_1}}{\ell(V)_1}$$

We can further decompose:

$$\ell(V) = \ell_{-\bar{1}} + \ell(V)_{\bar{0}} + \ell_{\bar{1}}$$

Let $e_1, \dots, e_m, e_{m+1}, \dots, e_{m+n}$ be a basis of $V = V_{\bar{0}} \oplus V_{\bar{1}}$

$\longrightarrow$ $a \in \ell(V)$ can be written as: $a = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$

where α is $(m \times m)$-matrix

δ is $(n \times n)$- "

β is $(m \times n)$ - "

γ is $(n \times m)$ - "

even elements have the form $\begin{bmatrix} \alpha & 0 \\ 0 & \delta \end{bmatrix}$

odd elements have the form $\begin{bmatrix} 0 & \beta \\ \gamma & 0 \end{bmatrix}$

$\rightarrow \quad l_{\bar 1} = \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix}$ and $l_{-\bar 1} = \begin{bmatrix} 0 & 0 \\ \gamma & 0 \end{bmatrix}$

$$\Rightarrow \Big[\underbrace{\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}}_{\text{even}}, \underbrace{\begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}}_{\text{odd}}\Big] = \underbrace{\begin{pmatrix} 0 & \alpha\beta - \beta\delta \\ 0 & 0 \end{pmatrix}}_{\text{odd}}$$

$$\Big[\underbrace{\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}}_{\text{even}}, \underbrace{\begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix}}_{\text{odd}}\Big] = \underbrace{\begin{pmatrix} 0 & 0 \\ \delta\gamma - \gamma\alpha & 0 \end{pmatrix}}_{\text{odd}}$$

$$\Big[\underbrace{\begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}}_{\text{odd}}, \underbrace{\begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix}}_{\text{odd}}\Big] = \underbrace{\begin{pmatrix} \beta\gamma & 0 \\ 0 & \gamma\beta \end{pmatrix}}_{\text{even}}$$

We call $l_{\bar 1}$ and $l_{\bar 1}$ $l_{\bar 0}$-modules

(form particular rep. under $l_{\bar 0}$)

$\rightarrow$ type 1 representation: $G_1 = G_{\bar 1} \oplus G_{-\bar 1}$

We can also have a type 2 rep. where G_1 is irreducible, i.e. one cannot set β or γ to zero and odd elements are of the form $\begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$

Type 2 classification:

$G = G_0 + G_1$	G_0	G_0 Rep. on G_1
$B(m,n)$	$\underset{\alpha}{B_m} + \underset{\delta}{C_n}$	vector × vector
$D(m,n)$	$\underset{\alpha}{D_m} + \underset{\delta}{C_n}$	vector × vector
$D(2,1,\alpha)$	$A_1 + A_1 + A_1$	vector × vector × vector
$F(4)$	$B_3 + A_1$	spinor × vector
$G(3)$	$G_2 + A_1$	spinor × vector
$Q(n)$	A_n	adjoint

convention: B_m is Lie-algebra of $SO(2m+1)$
C_n is Lie-alg of $Sp(2n)$

Type 1 classification:

G_0 acts on $G_{\bar{1}}$ and $G_{-\bar{1}}$ as irreducible representations which are contragradient ($\text{weights}(G_{-\bar{1}}) = -\text{weights}(G_{\bar{1}})$)

$G = G_0 + G_1$	G_0	G_0 rep on $G_{\bar{1}}$
$A(m,n)$	$A(m)+A(n)+C$	vector × vector × C
$A(m,m)$	$A_m + A_m$	vector × vector
$C(n)$	$C_{n-1} + C$	vector × C

where $\mathbb{C}$ denotes the abelian algebra of complex numbers

Matrix construction:

- $sl(m,n)$:
  $(m+n, m+n)$-matrices $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$
  of zero supertrace $str = tr\,\alpha - tr\,\delta$
- $osp(m,n)$:
  Define bi-linear form F with
  $F(x,y) = 0$ for $x \in V_{\bar 0}$, $y \in V_{\bar 1}$,
  $F(x,y) = F(y,x)$ for $x, y \in V_{\bar 0}$
  $F(x,y) = -F(y,x)$ for $x, y \in V_{\bar 1}$

then (for $s \in \mathbb{Z}_2$):

$$osp(m,n) = \left\{ a \in \ell(m,n) \,\middle|\, F(a(x),y) = -(-1)^{s(\deg x)} F(x, a(y)) \right\}$$

Then one has:

$$A(m,n) = sl(m+1, n+1) \quad (m \neq n,\ m,n \geq 0)$$
$$A(m,m) = sl(m+1, m+1)/I \quad (m \geq 0)$$
$$B(m,n) = osp(2m+1, 2n) \quad (m \geq 0 \quad n > 0)$$
$$C(n) = osp(2, 2n) \quad (n > 0)$$
$$D(m,n) = osp(2m, 2n) \quad (m \geq 2,\ n \geq 0)$$

Let's give an explicit description for the case $osp(2m+1, 2n)$:

$$F = \left(\begin{array}{ccc|cc} 0 & \mathbb{1}_m & 0 & & \\ \mathbb{1}_m & 0 & 0 & & \\ 0 & 0 & 1 & & \\ \hline & & & 0 & I_n \\ & & & -I_n & 0 \end{array}\right)$$

$\Rightarrow$ $a \in osp(m, n)$ is of the form

$$\left(\begin{array}{ccc|cc} a & b & u & x & x_1 \\ c & -a^T & v & y & y_1 \\ -v^T & -u^T & 0 & z & z_1 \\ \hline y_1^T & x_1^T & z_1^T & d & e \\ -y^T & -x^T & z^T & f & -d^T \end{array}\right)$$

$\rightarrow$ G_1 is irreducible rep. of G_0

For $osp(2m, 2n)$ we get

$$F = \left(\begin{array}{cc|cc} 0 & \mathbb{1}_m & & \\ \mathbb{1}_m & 0 & & \\ \hline & & 0 & \mathbb{1}_m \\ & & -\mathbb{1} & 0 \end{array}\right)$$

$\Rightarrow$ a is of same form as above with central row and column deleted

How do SCA's fit into this classification ?

We are searching superalgebras whose even part G_0 contains conformal group $SO(d,2) \to G_1$ should be spinor rep. of G_0

$B(m,n)$ and $D(m,n)$ have SO-subalg's <u>but</u> in vector-rep.

$F(4)$ has subalgebra $B_3 = SO(7)$, represented as spinor !

$\Rightarrow$ $F(4)$ is superconformal algebra !

R-symmetry : $A_1 = SU(2) = SO(3)$

Now, recall $\underset{\text{spinor}}{SO(5)} = \underset{\text{vector}}{Sp(2)} = C_2$

$\rightarrow$ $B(m,2)$ and $D(m,2)$ are superconformal algebras in $d=3$ with R-symmetry $SO(2m+1)$ and $SO(2m)$

$SO(6) = SU(4)$
spinor = vector

$\rightarrow$ $A(3,m)$ is SCA in $d=4$ with R-symmetry $A_m + C$, i.e.
$SU(m+1) \times U(1) = U(m+1)$

$SO(8)$ admits triality
$\rightarrow$ spinor representation is equivalent to vector representation

$\Rightarrow$ $B(4,n)$ is superconformal algebra in $d=6$ with R-symmetry
$C_n = Sp(n)$

same trick does <u>not</u> work for $SO(m)$ with $m > 8$
$\Rightarrow$ No superconformal algebras in $d>6$!!

<u>Explicitly</u> ($d=6$):
$SO(5,1)$ spinors are pseudoreal:

$$Q_{i\alpha} = \Omega_{ij}\left(C \sigma_0{}^T \right)_\alpha{}^\beta Q^\dagger_{j\beta}$$

$$S_{i\alpha} = \underset{\parallel}{\Omega_{ij}}\left(C \sigma_0{}^T \right)_\alpha{}^\beta S^\dagger_{j\beta}$$

$\mathrm{diag}(i\sigma_2, -i\sigma_2, \ldots, -i\sigma_2)$

$\mathrm{diag}(\underset{1}{-i\sigma_2}, \underset{2}{-i\sigma_2}, \dots, \underset{n}{-i\sigma_2})$

$C T^T C^{-1} = -T \,, \quad C^T = C \,, \quad C^* = C^{-1}$

Reality properties:

$$Q'_{i\alpha} = \Omega_{ij} \left(C\sigma_0^T\right)_\alpha{}^{\tilde\beta} S'^{\dagger}_{j\tilde\beta}$$

$$S'_{i\tilde\alpha} = \Omega_{ij} \left(C\bar\sigma_0^T\right)_{\tilde\alpha}{}^{\beta} Q'^{\dagger}_{j\beta}$$

SCA's exist only when all Q's have same chirality $\to P_+ = (1+\sigma_7)/2$

R-symmetry group is $Sp(n)$

relevant cases $n=1$, $n=2$.

<u>$n=1$</u>:

R-sym. is $Sp(1) = SU(2)$

$$[T_a, Q_i] = (-\sigma'_a/2)_i{}^j Q_j$$

$$[T_a, S_i] = (+\sigma'_a/2)_i{}^j S_j$$

$$[T_a, Q'_i] = (-\sigma'_a/2)_i{}^j Q'_j$$

$$[T_a, S'_i] = (-\sigma'_a/2)_i{}^j S'_j$$

$$\{Q_{i\alpha}, Q_{j\beta}\} = (P_+ \cancel{P} C)_{\alpha\beta}\, \varepsilon_{ij}$$

$$\{S_{i\alpha}, S_{j\beta}\} = (P_- \cancel{K} C)_{\alpha\beta}\, \varepsilon_{ij}$$

Ω_{ij} for $n=1$

$$\{Q_{i\alpha}, S_{j\beta}\} = \frac{\delta_{ij}}{2}\left[(M_{\mu\nu} P_+ \Gamma_\mu \Gamma_\nu C)_{\alpha\beta} + 2(P_+ C)_{\alpha\beta} D\right] - 8(i)\left(T_a \sigma_a/2\right)_{ij}\left(P_+ C\right)_{\alpha\beta}$$

<u>$n=2$:</u>

replace $T_a\, \sigma_a/2$ by $T_{ab} \left(\frac{1}{4}\right)\left(\frac{-i}{4}\right)[\Gamma'_a, \Gamma'_b]$

(recall vector$(Sp(2))$ = spinor$(SO(5))$)